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THE CONVEXITY OF THE FREE BOUNDARY FOR A PARABOLIC FREE BOUNDARY PROBLEM

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Abstract. In this paper, we study a parabolic free boundary problem which shows that the solutions of this free boundary problem are increasing functions. Furthermore, we provide a rigorous verification for that the free boundary for this problem is concave. As an application to the American option pricing problem, our results can be extended to consider the convexity of the early exercise boundary of an American call.

1. Introduction

In this paper, we consider the one phase one-dimensional free boundary problem for linear parabolic equations. Precisely, given a positive real-valued function $\psi \in C^3([0,\infty))$ with

$$\mathcal{L}_0\psi(x)\begin{cases} >0 & \text{for } 0\le x<d,\\ <0 & \text{for } d<x<\infty,\end{cases} \tag{1.1}$$

for some $d>0$, we shall discuss some properties of the solution (s,u) for

$$
\begin{aligned}
&(1.2) && \mathcal{L}u=0 && 0<x<s(t),\ 0<t<\infty,\\
&(1.3) && u(x,t)>\psi(x) && 0<x<s(t),\ 0<t<\infty,\\
&(1.4) && u(0,t)=\psi(0) && 0<t<\infty,\\
&(1.5) && u(x,0)=\psi(x) && 0\le x\le s(0), \qquad (\mathbf{P})\\
&(1.6) && u(s(t),t)=\psi(s(t)) && 0\le t<\infty,\\
&(1.7) && \frac{\partial u}{\partial x}(s(t),t)=\psi'(s(t)) && 0\le t<\infty,
\end{aligned}
$$

where

$$\mathcal{L}=\mathcal{L}_0-\frac{\partial}{\partial t}$$

and $\mathcal{L}_0$ is defined as

$$\mathcal{L}_0\equiv a(x)\frac{\partial^2}{\partial x^2}+b(x)\frac{\partial}{\partial x}+c(x).$$

We assume that coefficients $a,b,c\in C^{2+\alpha}([0,\infty))$ for some $\alpha\in(0,1)$ with $a(x)\ge a_0>0$ for $0\le x<\infty$ and $c(x)\le 0$ for $0\le x<\infty$. This problem had been studied

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by Kotlow [12]. He showed that the problem (P) is well-posed and that $u(x,t)$ and $s(t)$ are both nondecreasing functions of t. When $\mathcal{L}$ is given as

$$\mathcal{L}_0 \equiv \frac{\partial^2}{\partial x^2},$$

Problem (P) is called a Stefan problem. Friedman [8] showed that the free boundary for the Stefan problem is smooth and monotone increasing. Moreover, Friedman and Jensen [9] provided conditions under which the free boundary is a concave function by using the maximum principle to control the level curve.

By using the Feynman-Kac formula, the operator $\mathcal{L}$ in Problem (P) has a connection with the option pricing problem when the stock price is measured by time-homogeneous diffusions. Black and Scholes (BS) [1] suggested that the stock price can be measured by a geometric Brownian motion and derived a second order linear parabolic operator, namely BS pricing operator, for pricing an European option. In the BS framework, Merton [15] showed that the price of the American option together with an early exercise boundary is a solution of Problem (P).

Over the past two decades, a great deal of effort has been made on solving Problem (P) for pricing an American option. However, the exact solution of the American pricing model has not been found yet. Several researchers such as [2], [3], [6], [10], [11], [13] concentrated on finding more accurately expansions or simulations for the early exercise boundary. By observing the results of simulations, people are convinced that $s(t)$ is convex for the American put and is concave for the American call. The rigorous verifications of the convexity for $s(t)$ of the American put have been proposed by Chen *et al.* [4] and Ekström [5] during 2004. The convexity of s for the American put provides a useful information for its approximation. When the remaining time is close to zero, Chen *et al.* [4] used this information to provide an asymptotic formula for the early exercise boundary of the American put.

To show the convexity of $s(t)$ for the American put, Chen *et al.* [4] and Ekström [5] defined $s(t) = Ke^{z(\tau)}$ and showed that $z(\tau)$ is a convex function by using the same argument of [9], where $\tau = T - t$. Consequently, $s(t) = Ke^{z(\tau)}$ is a convex function since e^x is a convex function. However, their methods do not work for verifying the concavity of $s(t)$ for the American call. Although their methods can show that $z(\tau)$ is a concave function for the American call, we still do not know the concavity of $s(t) = Ke^{z(\tau)}$. So far, we have not seen any published works on the concavity of $s(t)$ for the American call.

In this paper, we make the following assumptions which will be used later.

Assumptions

(A) $c(x)$ is a nonincreasing function in $[0, \infty)$.
(B) $\psi(x)$ is a positive strictly increasing convex function in $[0, \infty)$.
(C) $\frac{d}{dx}\mathcal{L}_0\psi(x) \le 0$ and $c(x) + b'(x) \le 0$ in $[0, \infty)$.
(D) $\limsup_{\xi\to\infty} \mathcal{L}_0\psi(\xi) < 0$.

Under the given assumptions, we shall show that $s(t)$, $u(\cdot, t)$ and $u(x, \cdot)$ are increasing functions . The main contribution of this paper is to provide a rigorous verification of the concavity of $s(t)$. The obtained results can be applied to consider the convexity of the early exercise boundary in American option pricing problems.

The content of this paper is organized as follows. In section 2, we provide some useful properties for the solution of Problem (P). The concavity of the free boundary is obtained in section 3. Finally, a concise conclusion is given in section 4.

2. Properties of the Solution

Let $\{s, u\}$ be the solution of (P) and denote C, namely continuation region, as

$$C = \{(x,t); 0 < x < s(t),\ 0 < t < \infty\}. \tag{2.1}$$

Now we define $\hat{u}$ on $\bar{Q}$ by

$$\hat{u}(x,t) = \begin{cases} u(x,t) & \text{if } (x,t) \in C, \\ \psi(x) & \text{if } (x,t) \in \bar{Q} - C, \end{cases}$$

where $\bar{Q} = (0,\infty) \times (0,\infty)$.

In this section, we will show that

(1) $s(t)$ is a strictly increasing function with $s(0) = d$,
(2) $u(x,\cdot)$ is a strictly increasing function in C,
(3) $u(\cdot,t)$ is an increasing function in C.

In order to prove these results, we need some preliminaries.

Definition 2.1. $\{s,u\}$ is a solution of (1.2)-(1.7) if

(i) $s \in C^{\gamma}([0,\infty))$ for some $\gamma \in [1/2, 1)$ and is Lipschitz continuous for $0 < t < \infty$.
(ii) $u \in C^{2,1}(C) \cap C^0(\bar{C})$.
(iii) $\frac{\partial u}{\partial x}$ is bounded and has a continuous extension to $\bar{C}$.
(iv) $\frac{\partial^2 u}{\partial x^2}$ is bounded and has a continuous extension to $\bar{C} - (0,0) - (s(0),0)$.
(v) Conditions (1.3)-(1.7) are satisfied.

Definition 2.2. Given $t \in [0,\infty)$, the t-section of C is defined to be

$$C_t = \{x \in \mathbb{R} | 0 < x < s(t)\}. \tag{2.2}$$

Clearly, we have

$$C = \bigcup_{t<\infty} (C_t \times \{t\})$$

and

$$s(t) = \sup\{x | x \in C_t\}. \tag{2.3}$$

The following properties of $u(x,t)$ and $s(t)$ have been proved by Kotlow [12].

Lemma 2.3. *Let $\{s,u\}$ be a solution of (P). They have the following properties:*

(a) $u_t > 0$ *in* C.
(b) $s(0) = d$ *and* $s(t) > d$ *for* $0 < t < \infty$.
(c) $s(t)$ *is a nondecreasing function.*
(d) *There exists a* $s^{\infty} \in (d,\infty)$ *such that* $s(t) \to s^{\infty}$ *uniformly as* $t \to \infty$ *if (D) holds.*

Lemma 2.4. *Let $\{s,u\}$ be a solution of (P). At points $(s(t),t)$, $t > 0$, u satisfies $u_t(s(t),t) = 0$.*

Proof. By (1.6), we have $u(s(t),t) = \psi(s(t))$. Differentiating $u(s(t),t) = \psi(s(t))$ with respect to t, we have

$$u_x(s(t),t)s'(t) + u_t(s(t),t) = \psi'(s(t))s'(t)$$

for each $t > 0$, where $s(t)$ is differentiable. Since $u_x(s(t),t) = \psi'(s((t))$ by (1.7), we have $u_t(s(t),t) = 0$ for almost every $t > 0$. By (iv) of Definition 2.1, we have $u_t(s(t),t) \equiv 0$. □

This proof is proposed by Kotlow [12].

Lemma 2.5. *Let $\{s,u\}$ be a solution of (P) and $w = \hat{u} - \psi$ in $\bar{Q}$. Then $w(\cdot,t)$ has a local maximum in $(0,s(t))$. Moreover, this local maximum can not lie in $(d,s(t))$.*

Proof. By (1.3) and (1.6), we have $w(0,t) = w(s(t),t) = 0$ and $w(x,t) > 0$ on C. This implies that there exists a $d' \in (0,s(t_0))$ for some $t_0 > 0$, which may depend on t_0, such that $w(d',t_0)$ is a local maximum. Now, we claim that $d' \notin (d,s(t_0))$. Suppose that $d' \in (d,s(t_0))$ is a local maximum of $w(x,t_0)$. We define

$$\begin{aligned} \Omega_{t_0} &= \{(x,t) \in C | d \le x \le s(t),\ t \le t_0\}, \\ \partial_p \Omega_{t_0} &= \{(x,t) \in \partial\Omega_{t_0} | t < t_0\}. \end{aligned}$$

and apply the differential operator $\mathcal{L}$ to w. By (1.1) and (1.2), w satisfies the parabolic differential equation

$$\mathcal{L}w = (\mathcal{L}_0 - \frac{\partial}{\partial t})u - \mathcal{L}_0\psi(x) = -\mathcal{L}_0\psi(x) > 0 \quad \text{on } \Omega_{t_0}.$$

Now, we apply the maximum principle to this equation. Since $(d',t_0) \in \Omega_{t_0} - \partial_p\Omega_{t_0}$ is a nonnegative maximum over Ω_{t_0}, it implies that w is a constant function on Ω_{t_0}. This contradicts to that $w_t = u_t > 0$. So $d' \le d$. □

Definition 2.6. Let

$$\mathcal{M}_0 = a(x)\frac{\partial^2}{\partial x^2} + (b(x) + a'(x))\frac{\partial}{\partial x} + (c(x) + b'(x)) \tag{2.4}$$

be an elliptic operator. We define a parabolic operator $\mathcal{M}$ as

$$\mathcal{M} = \mathcal{M}_0 - \frac{\partial}{\partial t}.$$

Theorem 2.7. *Let $\{s,u\}$ be a solution of (P). Then*

(a) *$s(t)$ is a strictly increasing function.*
(b) *$u_x(x,t) > 0$ for $(x,t) \in C$ if (A), (B) hold.*
(c) *$u_x(x,t) < \psi'(x)$ for $(x,t) \in C_d$, where $C_d = \{(x, t \in C | x > d)\}$, if (A), (B), (C) hold.*

Proof. For (a), we only need to show that $s(t_2) \ne s(t_1)$, for $t_2 < t_1$ by (c) in Lemma 2.3. Suppose that there is an interval $[t_1,t_2]$ such that $s(t) = s(t_1)$ for all $t \in [t_1,t_2]$, then $u_x(s(t),t)$ is a constant function for all $t \in (t_1,t_2)$. Since $\mathcal{L}u = 0$ in $(0,s(t_1)) \times (t_1,t_2)$ and $u(s(t),t) = \psi(s(t_1))$ for all $t \in [t_1,t_2]$, we have $u \in C^\infty([0,s(t_1)) \times (t_1,t_2))$. Since $u_t(s(t_1),t_1) = 0$ for $t \in (t_1,t_2)$, $u_t > 0$ in $(0,s(t_1)) \times (t_1,t_2)$ and $\mathcal{L}u_t = 0$ in $(0,s(t_1)) \times (t_1,t_2)$, we have $u_{tx}(s(t),t) < 0$ for $t \in (t_1,t_2)$ by applying the boundary point form of the maximum principle. And then, we have $u_{xt}(s(t),t) = u_{tx}(s(t),t) < 0$ which implies that $u_x(s(t),t)$ is strictly decreasing for $t \in (t_1,t_2)$. On the other hand, $s(t)$ is a nondecreasing function in $[t_1,t_2]$ and $\psi(x)$ is a strictly increasing convex function. This implies that $\psi'(s(t))$ is a nondecreasing function in $[t_1,t_2]$. Hence, we obtain that $u_x(s(t),t) = \psi'(s(t))$ is a nondecreasing function in $[t_1,t_2]$ and there is a contradiction. So $s(t)$ is a strictly increasing function.

We now show that (b) $u_x > 0$. We first consider that

$$\begin{aligned} \frac{\partial}{\partial t}u_x &= \frac{\partial}{\partial x}\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(\mathcal{L}_0 u) \\ &= \frac{\partial}{\partial x}(a(x)u_{xx} + b(x)u_x + c(x)u) \\ &= a(x)\frac{\partial^2}{\partial x^2}u_x + (b(x) + a'(x))\frac{\partial}{\partial x}u_x + (c(x) + b'(x))u_x + c'(x)u \end{aligned}$$

Consequently, u_x satisfies the parabolic differential equation

$$\mathcal{M}u_x = -c'(x)u.$$

Let $w(x,t) = \hat{u}(x,t) - \psi(x)$ on $\bar{Q}$. Then $w(0,t) = 0$ and $w(x,t) > 0$ on C by (1.3) and (1.4). So we have

$$w_x(0,t) = \lim_{h \to 0^+} \frac{w(h,t) - w(0,t)}{h} \geq 0.$$

This implies that

$$u_x(0,t) \geq \psi'(0) > 0 \tag{2.5}$$

for $0 < t < \infty$. We apply the maximum principle to this equation. Since $u_x(s(t),t) = \psi'(s(t)) > 0$ for $t \geq 0$ by (1.7), $u_x(0,t) > 0$ for $0 < t < \infty$ by (2.5), $u_x(x,0) = \psi'(x) > 0$ for $0 < x < s(0)$ by (1.5) and $c'(x) \leq 0$ by assumption (A), this shows $u_x(x,t) > 0$ for $0 < x < s(t)$ and $0 < t < \infty$.

Finally, we show that (c) $u_x(x,t) < \psi'(x)$ for $d < x < s(t)$ and $0 < t < \infty$. Let $w(x,t) = \hat{u}(x,t) - \psi(x)$ on $\bar{Q}$. Since $w(s(t),t) = w_x(s(t),t) = 0$ by (1.6) and (1.7), $w(x,t) > 0$ on C by (1.3), and the continuity of w and w_x, we have that, for any $t > 0$, there exists a $\delta > 0$ such that $w_x(x,t) < 0$ for $s(t) - \delta < x < s(t)$ and $t > 0$. Suppose that there is a $x' \in (d, s(t) - \delta)$ with $w_x(x',t) \geq 0$, then there exists a local maximum in $[x', s(t)) \subseteq (d, s(t))$. This contradicts Lemma 2.5. So we obtain that there is no $x \in (d, s(t))$ with $w_x(x,t) \geq 0$. Hence, we have $w_x(x,t) < 0$ for $(x,t) \in C_d$. This implies that $u_x(x,t) < \psi'(x)$ for $(x,t) \in C_d$. □

3. Concavity of the free boundary

Suppose that (A), (B) and (C) hold. We will show that $s(t)$ in (2.1) is a concave function; consequently the continuation region of C in (2.1) is a convex set.

In order to prove our main theorem, we need the following lemma.

Lemma 3.1. *Let $\{s,u\}$ be a solution of (P) and define $w = u - \psi$ on $\bar{C}_d$, where $\bar{C}_d = \{(x,t) \in \mathbb{R}^2 | d < x \leq s(t), 0 < t < \infty\}$. At points $(s(t),t)$, $t > 0$, w has the following properties*

$$\begin{aligned} &w(x,t) = 0,\ w_t(s(t),t) = 0,\ w_x(s(t),t) = 0, \\ &\lim_{x \to s(t)} w_{xx}(x,t) = -\frac{1}{a(s(t))}(\mathcal{L}_0\psi(s(t))), \\ &\lim_{x \to s(t)} w_{tx}(x,t) = \frac{1}{a(s(t))}(\mathcal{L}_0\psi(s(t)))s'(t), \\ &\lim_{x \to s(t)} w_{tt}(x,t) = -\frac{1}{a(s(t))}(\mathcal{L}_0\psi(s(t)))(s')^2(t), \end{aligned}$$

where $s(t)$ is differentiable.

Proof. Since $u(s(t),t)=\psi(s(t))$, $u_x(s(t),t)=\psi'(s(t))$ and $u_t(s(t),t)=0$, we obtain the first three equalities. By (1.1) and (1.2), we have

$$aw_{xx}+bw_x+cw-w_t=-\mathcal{L}_0\psi.$$

Since $w=0$, $w_x=0$ and $w_t=0$ at $(s(t),t)$, we have

$$w_{xx}=-\frac{1}{a}(\mathcal{L}_0\psi).$$

Differentiating the equality

$$w_x(s(t),t)=0$$

with respect to t, we have

$$w_{xx}s'(t)+w_{tx}=0.$$

So $w_{tx}=-w_{xx}s'(t)=\frac{1}{a}(\mathcal{L}_0\psi(x))s'(t)$. Next, w_{tt} can be found by differentiating

$$w_t(s(t),t)=0.$$

We obtain

$$w_{xt}(s(t),t)s'(t)+w_{tt}(s(t),t)=0.$$

So $w_{tt}=-w_{xt}(s(t),t)s'(t)=-\frac{1}{a}(\mathcal{L}_0\psi(x))(s')^2(t)$. □

Now, we can state and proof our main theorem of this paper.

Theorem 3.2. *Let (A), (B), (C) and (D) hold, $\mathcal{M}_0\psi'\le 0$ and $\{s,u\}$ be a solution of (P). Then $s(t)$ is a concave function.*

Proof. We have known that $s(t)$ is a strictly increasing function. Suppose that there is an interval I such that $s(t)$ is a convex function in the interval I. There is a $t_0\in I$ with $s'(t_0)=m>0$ since $s(t)$ is a strictly increasing function and is differentiable almost everywhere. Then, $s'(t)\ge m$ for almost every $t>t_0$ in I. We consider the line

$$y(t)=m(t-t_0)+s(t_0)$$

for some $t>0$. So $y(t_0)=s(t_0)$. Since $s(t)$ is bounded above by (d) of Lemma 2.3 and $m>0$, there must exist another point $t_1>t_0$ such that $y(t_1)=s(t_1)$. Let $f(t)=w(y(t),t)$ for some $t>t_2$ where $t_2=\inf\{t|(y(t),t)\in C_d\}$. Since $w_t(s(t),t)=0$ and $w_x(s(t),t)=0$, we have

$$\begin{aligned} f'(t_i) &= mw_x(y(t_i),t_i)+w_t(y(t_i),t_i)\\ &= mw_x(s(t_i),t_i)+w_t(s(t_i),t_i)=0,\ i=0,1. \end{aligned}$$

We also have $f(t_0)=w(y(t_0),t_0)=0$, $f(t_1)=w(y(t_1),t_1)=0$ and $(y(t),t)\in C_d$ for $t\in(t_0,t_1)$, which implies that $f(t)=w(y(t),t)>0$ for $t\in(t_0,t_1)$. So there exists a local maximum of f between t_0 and t_1, namely $f(t_3)$ where $t_3\in(t_0,t_1)$. This implies that $f(t_3)>0$ and $f'(t_3)=0$. Since $w=u-\psi$ is a solution of parabolic equation and $f(t)=w(y(t),t)$, which does not oscillate as $t\to t_1$. This implies that $f(t)$ do not produce an infinite sequence of local maximum whose locations tends to t_1. Consequently there is no infinite sequence of local maximum whose locations tends to t_1. Therefore, we can assume that t_3 is the first one from t_1 and there is no local maximum between t_3 and t_1. Since $f(t_0)=f(t_1)=0$, $f(t_3)>0$, and $f'(t_i)=0$, $i=0,1,3$, we have

$$f'(t)<0 \text{ for } t\in(t_3,t_1) \tag{3.1}$$

and $f'(t)>0$ for some interval, say (t_4,t_3), where $t_0\le t_4<t_3$.

Since

$$\lim_{x\to s(t)}\frac{w_t(x,t)}{w_x(x,t)}=-s'(t)$$

by l'Hôpital's rule and $w>0$ on C_d, let

$$v=\begin{cases}\frac{w_t}{w_x} & \text{if } (x,t)\in C_d,\\ -s'(t) & \text{if } x=s(t),\end{cases}$$

which is well-defined on $\bar{C}_d$. Then we have that

$$\begin{aligned} f'(t) &= mw_x(y(t),t)+w_t(y(t),t)\\ &= w_x(y(t),t)(m+v(y(t),t))\end{aligned} \tag{3.2}$$

for $t>t_2$. Applying the differential operator $\mathcal{L}$ to the equality $vw_x=w_t$, we obtain that v satisfies the following parabolic equation

$$av_{xx}+(b+\frac{2w_{xx}}{w_x})v_x-\frac{1}{w_x}(\mathcal{M}_0\psi')v-v_t=0 \text{ on } \bar{C}_d. \tag{3.3}$$

Let Γ_α be the level curves on which $v=\alpha$. Since $w_x<0$, $\mathcal{M}_0\psi'\le 0$ and v satisfies the parabolic equation (3.3), the x-coordinate along Γ_α can not first increase (decrease) and then decrease (increase) by using the extensions of maximum principle (see Appendix A). This is because that there would be a region whose parabolic boundary is a part of Γ_α; consequently $v\equiv\alpha$ in this region and hence $v\equiv\alpha$ in C_d. Since Γ_α is understood to be continued as long it remains in C_d, for each α there is a $g_\alpha(t)$ such that

$$\Gamma_\alpha=\{(g_\alpha(t),t)|v(g_\alpha(t),t)=\alpha,t>0\}.$$

Since $f'(t_i)=0$, $i=0,1,3$ and $f'(t)=w_x(y(t),t)(m+v(y(t),t))$, we have $v(y(t_i),t_i)=-m$, $i=0,1,3$. This implies that $(y(t_i),t_i)\in\Gamma_{-m}$, $i=0,1,3$. Now we consider the function $g_{-m}(t)$. Since $(y(t_i),t_i)\in\Gamma_{-m}$, that is $v(y(t_i),t_i)=-m$, $i=0,1,3$, we have $y(t_i)=g_{-m}(t_i)$, $i=0,1,3$.

Since $g_{-m}(t)$ is continuous on (t_2,t_1), we only have the following two cases: (1) $y(t)<g_{-m}(t)$ for $t\in(t_3,t_1)$, and (2) $y(t)>g_{-m}(t)$ for $t\in(t_3,t_1)$.

We first consider case (1). We have $w_x<0$ on C_d by (c) of Theorem 2.7. Since $f'(t)=w_x(y(t),t)(m+v(y(t),t))<0$ for $t\in(t_3,t_1)$ by (3.1) and (3.2), this implies that

$$v(y(t),t)>-m, \text{ for } t\in(t_3,t_1). \tag{3.4}$$

Since $g_{-m}(t_1)=y(t_1)=s(t_1)$ and $y(t)<g_{-m}(t)<s(t)$ for $t\in(t_3,t_1)$, there is a $\delta>0$ such that $y'(t)>g'(t)>s'(t)$ for $t\in(t_1-\delta,t_1)$. Since $y'(t)=m$, we have $v(s(t),t)=-s'(t)>-y'(t)=-m$ for $t\in(t_1-\delta,t_1)$. Let $\Omega=\{(x,t)|y(t)\le x\le s(t),\ t_1-\delta\le t\le t_1\}$. Since $v(g_{-m}(\tau),\tau)=-m$ for $\tau\in(\tau,\tau_1+\delta)$, the negative minimum of v in Ω which is attained at (x',t') is less than $-m$. By Theorem A.1 which is displayed in Appendix A, we have

$$v(x,t')=v(x',t')<-m,\ \forall x \text{ such that } (x,t)\in\Omega,$$

which contradicts to $v(s(t'),t')>-m$ and $v(y(t'),t')>-m$.

Now we consider case (2). We have known that the level curves Γ_α of a parabolic equation is continuous. We consider the line $y(t)$ for $t\in(t_3,t_1)\cup(t_2,t_0)$. Since (3.4) is also true for case (2), we have $v(y(t),t)>-m$ for $t\in(t_3,t_1)$. And then, we also have $f(t_0)=0$ and $f(t)>0$ for $t\in(t_2,t_0)$. This implies that there is a $\delta_2>0$ such that $f'(t)<0$ for $t\in(t_0-\delta_2,t_0)$. Since $f'(t)=w_x(y(t),t)(m+v(y(t),t))$ and

$f'(t) < 0$ for $t \in (t_0-\delta_2, t_0)$, we have $v(y(t), t) > -m$ for $t \in (t_0-\delta_2, t_0)$. Now we can select a suitable $\delta > 0$ such that $v(y(t), t) > -m$ for $t \in (t_0-\delta, t_0) \cup (t_1-\delta, t_1)$. Since $v(y(t_0), t_0) = -m = v(y(t_1), t_1)$ and $v(y(t), t) > -m$ for $t \in (t_0 - \delta, t_0) \cup (t_1 - \delta, t_1)$, there exists a $t' \in (t_0 - \delta, t_0)$ and a $t'' \in (t_1 - \delta, t_1)$ such that

$$v(y(t'), t') = \beta = v(y(t''), t''), \text{ for aome } \beta > -m.$$

Since the level curves of a parabolic equation is continuous, there is a level curve Γ_β connecting $(y(t'), t')$ and $(y(t''), t'')$. On the other hand, we have $(y(t''), t'') \in \Omega_1$, where $\Omega_1 = \{(x, t) \in C | g_{-m}(t) \leq x < s(t),\ t_0 < t < t_1\}$. This contradicts to that $\Gamma_{-m} \cap \Gamma_\beta \neq \emptyset$. So case (2) does not hold.

Since both case (1) and case (2) do not hold, we conclude that there is no such point t_0 with $s''(t) > 0$ for $t > 0$. Thus, $s(t)$ is a concave function. □

Remark 3.3. Given $\alpha \in R$ and $g_\alpha(t)$ be the function such that

$$v(g_\alpha(t), t) = \alpha$$

with $g_\alpha(t_0) = s(t_0)$, where $v(s(t_0), t_0) = \alpha$. Then

$$\frac{dv}{dt} = v_x \frac{dg_\alpha(t)}{dt} + v_t = 0.$$

By Sard's lemma, the set of $v_x(x, t) = 0$ has measure zero. Hence $-\frac{v_t}{v_x}$ is defined for almost every point on Ω. We consider the following IVP

$$\frac{dg_\alpha(t)}{dt} = -\frac{v_t}{v_x} \quad (a.e.) \tag{3.5}$$

with $g_\alpha(t_0) = s(t_0)$. Indeed, the weak solution for (3.5) exists. Therefore $g_\alpha(t)$ is continuous for all t with $v(g_\alpha(t), t) = \alpha$.

The detail about the extensions of maximum principle can be found in Friendman [7].

Remark 3.4. Let $\psi(x)$ denote the final payoff function and suppose that the stock price is measured by the time-homogeneous diffusions

$$dX(t) = r(X(t))dt + \sigma(X(t))dB(t), \text{ and } X(s) = x.$$

By using the Feynman-Kac formula, we have that the European option's price v under the constant interest rate $r > 0$ is a solution of the following initial value problem

$$\mathcal{L}v = 0, \text{ with } v(x, 0) = \psi(x),$$

where $\tau = T - t$, $\mathcal{L} = \mathcal{L}_0 - \frac{\partial}{\partial \tau}$ and

$$\mathcal{L}_0 = \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2} + r(x)\frac{\partial}{\partial x} - r. \tag{3.6}$$

However, an American option gives the holder a right to exercise prior the maturity. For the possibility of early exercise, the value of an American option is no longer less then its immediate exercise value, that is $v(x, t) \geq \psi(x)$. Now, we separate the domain $\{(x, t) | 0 \leq x < \infty,\ 0 \leq t \leq T\}$ into two parts: (i) a continuation region $\mathcal{C} = \{(x, t) | v(x, t) > \psi(x)\}$, and (ii) a stopping region $\mathcal{S} = \{(x, t) | v(x, t) = \psi(x)\}$. Given $t \in (0, T)$, when $\psi(x)$ is a strictly increasing function, there is a time-depended function $s(t)$ such that $v(x, t) > \psi(x)$ for $0 < x < s(t)$ (see [3]). Hence, the continuation region is described as $\mathcal{C} = \{(x, t) | 0 < x < s(t),\ 0 < t < T\}$. On the other hand, given $t \in (0, T)$, when $\psi(x)$ is a strictly decreasing function, there

is a time-depended function $s(t)$ such that $v(x,t) > \psi(x)$ for $s(t) < x < \infty$. Hence, the continuation region is described as $\mathcal{C} = \{(x,t)|s(t) < x < \infty,\ 0 < t < T\}$. And then, by using the no arbitrage condition, the price of the American option satisfies the high contact condition, that is $v_x(x,t) = \psi'(x)$. Therefore, the American option's price v with an early exercise boundary s is a solution of Problem (P), where the operator $\mathcal{L}_0$ is given by (3.6). Consequently, the rigorous verification of the concavity for the early exercise boundary can be obtained by applying the presented results.

4. Conclusion

In this paper, we consider a parabolic free boundary problem. Under given assumptions, we have shown that the solutions of this problem are strictly increasing. Moreover, we have obtained that the free boundary $s(t)$ is a concave function. The American call pricing model derived by Merton [15] is a specific case of this parabolic free boundary problem. Nevertheless, the rigorous verification of the concavity for the early exercise boundary has been neglected by their works. By using our results, a rigorous verification of the concavity of the early exercise boundary can be obtained.

Appendix A. Extensions of maximum principle

In this section, we display the extensions of maximum principle in [7]. Let D be a 2-dimensional domain bounded by an open interval B on $t = 0$, an open interval B_T on $t = T$ and two continuous curves $C_i : x = \gamma_i(t)$, $i = 1, 2$ defined for $0 < t < T$. For any $P^0 = (x^0, t^0)$ in D, we denote $C(P^0)$ as the component of $D \cap \{t = t^0\}$ which contains P^0.

Consider the differential operator

$$\mathcal{L}u \equiv a(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x} + c(x)u + d(x)\frac{\partial}{\partial t}$$

with continuous coefficients in domain D. Assume that (A) $\mathcal{L}$ is parabolic in D; (B) the coefficients of $\mathcal{L}$ are continuous functions in D; (C) $C(x) \le 0$ in D. Here, Friedman [7] did not make the assumption of $d(x)$.

Theorem A.1. *Let (A), (B) and (C) hold. If $\mathcal{L}u \ge 0$ ($\mathcal{L}u \ge 0$) in D and if u has a positive maximum (negative minimum) in D which is attained at a point $P^0 = (x^0, t^0)$, then $u(P) = u(P^0)$ for all $P \in C(P^0)$.*

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